An example in calculating Arc Length

Yue Kwok Choy

(1) Formulas

Let $y = f(x)$ be differentiable function, arc length $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

If $x = h(y)$, $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

For parametric equation: $x = u(t), y = v(t), \alpha \le t \le \beta$, $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

For polar equation: $r = f(\theta), \alpha \le t \le \beta$, $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

(2) Example

Find the arc length of the parabola $\mathbf{y^2 = x}$ from $(\mathbf{0}, \mathbf{0})$ to $(\mathbf{1}, \mathbf{1})$.

This problem looks easy, but it needs quite a lot of integration techniques. You need to pay more attention on how to handle the problem rather than the tedious calculations in the followings.

Method 1

$y^2 = x \Longrightarrow \frac{dx}{dy} = 2y$

Arc length , $L = \int_0^1 \sqrt{1 + (2y)^2}\, dy$

Let $2y = \tan\theta$, $dy = \frac{1}{2}\sec^2\theta d\theta$.

When $y = 0, \theta = 0$. When $y = 1, \theta = \tan^{-1}2$.

$$L = \int_0^{\tan^{-1}2} \sqrt{1 + (\tan\theta)^2}\left(\frac{1}{2}\sec^2\theta d\theta\right) = \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta d\theta = \frac{1}{2}\int_0^{\tan^{-1}2} \sec\theta\, d(\tan\theta)$$

$$= \left[\frac{1}{2}\sec\theta\tan\theta\right]_0^{\tan^{-1}2} - \frac{1}{2}\int_0^{\tan^{-1}2} \tan\theta\, d(\sec\theta) = \frac{1}{2}(\sqrt{5})(2) - \frac{1}{2}\int_0^{\tan^{-1}2} \tan^2\theta\sec\theta\, d\theta$$

$$= \sqrt{5} - \frac{1}{2}\int_0^{\tan^{-1}2} (\sec^2\theta - 1)\sec\theta\, d\theta = \sqrt{5} - \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta d\theta + \frac{1}{2}\int_0^{\tan^{-1}2} \sec\theta\, d\theta$$

$$= \sqrt{5} - L + \frac{1}{2}\int_0^{\tan^{-1}2} \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta = \sqrt{5} - L + \frac{1}{2}\int_0^{\tan^{-1}2} \frac{d(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}$$

$$= \sqrt{5} - L + \left[\frac{1}{2}\ln(\sec\theta + \tan\theta)\right]_0^{\tan^{-1}2}$$

$$2L = \sqrt{5} + \frac{1}{2}\ln(\sqrt{5} + 2)$$

$$L = \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(\sqrt{5} + 2)\right] \approx \mathbf{1.4789428575446}$$

Method 2

For those who understand ***hyperbolic functions***, the calculations are easier.

Use the substitution $2y = \sinh\theta$. $dy = \frac{1}{2}\cosh\theta d\theta$.

Arc length , $L = \int_0^1 \sqrt{1+(2y)^2}\,dy = \int_0^{\sinh^{-1}(2)} \sqrt{1+(\sinh\theta)^2}\left(\frac{1}{2}\cosh\theta d\theta\right)$

$$= \frac{1}{2}\int_0^{\sinh^{-1}(2)} \cosh^2\theta d\theta = \frac{1}{2}\int_0^{\sinh^{-1}(2)} \frac{1+\cosh 2\theta}{2} d\theta = \frac{1}{4}\left[\theta + \frac{\sinh 2\theta}{2}\right]_0^{\sinh^{-1}(2)}$$

$$= \frac{1}{4}\left[\theta + \frac{2\sinh\theta\cosh\theta}{2}\right]_0^{\sinh^{-1}(2)} = \frac{1}{4}\left[\theta + \sinh\theta\sqrt{1+(\sinh\theta)^2}\right]_0^{\sinh^{-1}(2)}$$

$$= \frac{1}{4}\left[\sinh^{-1}(2) + (2)(\sqrt{5})\right] = \boldsymbol{\frac{1}{4}\left[2\sqrt{5} + \sinh^{-1}(2)\right] \approx 1.4789428575446}$$

I love this method, short and nice!

Method 3

$y^2 = x \Rightarrow y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ (Note that we only need the positive branch of the parabola.)

$L = \int_0^1 \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = \int_0^1 \sqrt{1+\frac{1}{4x}}\,dx = \int_0^1 \sqrt{\frac{4x+1}{4x}}\,dx$

Let $t = \sqrt{\frac{4x+1}{4x}} \Rightarrow x = -\frac{1}{4\,(1-t^2)}, t \neq -1$ and $t \neq 1$

$dx = -\frac{t}{2(t^2-1)^2}dt$

When $x = 0, t = \infty$. When $x = 1$, $t = \frac{\sqrt{5}}{2}$

$L = \int_\infty^{\frac{\sqrt{5}}{2}} t\left(-\frac{t}{2(t^2-1)^2}dt\right) = -\frac{1}{8}\int_\infty^{\frac{\sqrt{5}}{2}} \frac{4t^2}{(t^2-1)^2}dt$

We have to express the integrand in ***partial fractions***, observe that:

$$\frac{4t^2}{(t^2-1)^2} = \left(\frac{2t}{t^2-1}\right)^2 = \left(\frac{1}{t+1}+\frac{1}{t-1}\right)^2 = \frac{1}{(t+1)^2}+\frac{1}{(t-1)^2}+\frac{2}{(t+1)(t-1)} = \frac{1}{(t+1)^2}+\frac{1}{(t-1)^2}+\frac{1}{t-1}-\frac{1}{t+1}$$

$$L = -\frac{1}{8}\int_\infty^{\frac{\sqrt{5}}{2}} \left[\frac{1}{(t+1)^2}+\frac{1}{(t-1)^2}+\frac{1}{t-1}-\frac{1}{t+1}\right]dt = \frac{1}{8}\left[\frac{1}{t+1}+\frac{1}{t-1}-\ln\left(\frac{t-1}{t+1}\right)\right]_\infty^{\frac{\sqrt{5}}{2}}$$

$$= \frac{1}{8}\left[\frac{1}{\frac{\sqrt{5}}{2}+1}+\frac{1}{\frac{\sqrt{5}}{2}-1}-\ln\left(\frac{\frac{\sqrt{5}}{2}-1}{\frac{\sqrt{5}}{2}+1}\right)\right] \approx \mathbf{1.4789428575446}$$

Method 4

$$y^2 = x \Longrightarrow y = \sqrt{x} \Longrightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = \int_0^1 \sqrt{\frac{4x+1}{4x}}\, dx$$

Consider $\quad I = \frac{1}{2}\int \sqrt{\frac{4x+1}{x}}\, dx.\quad$ Put $\; u = \frac{4x+1}{x},\quad du = -\frac{1}{x^2}\, dx$

$$I = \frac{1}{2}\int \sqrt{\frac{4x+1}{x}}\, dx = \frac{1}{2}\left[x\sqrt{\frac{4x+1}{x}} - \int x\; d\sqrt{\frac{4x+1}{x}}\right] = \frac{1}{2}\left[x\sqrt{\frac{4x+1}{x}} - \int x \left(\frac{1}{2\sqrt{\frac{4x+1}{x}}}\right)\left(-\frac{1}{x^2}\, dx\right)\right]$$

$$= \frac{1}{2}\left[x\sqrt{\frac{4x+1}{x}} + \frac{1}{2}\int \frac{1}{\sqrt{x(4x+1)}}\, dx\right] = \frac{1}{2}\left[x\sqrt{\frac{4x+1}{x}} + \frac{1}{4}\int \frac{1}{\sqrt{x^2+\frac{x}{4}}}\, dx\right] = \frac{1}{2}\left[x\sqrt{\frac{4x+1}{x}} + \frac{1}{4}\int \frac{1}{\sqrt{\left(x+\frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2}}\, dx\right]$$

The last integral is a standard integral $\; E = \int \frac{dx}{\sqrt{x^2-a^2}}\;$ which can be solved easily as:

Put $\; u^2 = x^2 - a^2 \Longrightarrow 2udu = 2xdx \Longrightarrow \frac{dx}{u} = \frac{du}{x} = \frac{dx+du}{u+x} = \frac{d(x+u)}{x+u}$

$$E = \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{dx}{u} = \int \frac{d(x+u)}{x+u} = \ln{(x+u)} + c = \ln\left(x + \sqrt{x^2-a^2}\right) + c$$

Therefore,

$$I = \frac{1}{2}\left[x\sqrt{\frac{4x+1}{x}} + \frac{1}{4}\ln\left[\left(x+\frac{1}{8}\right) + \sqrt{\left(x+\frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2}\right]\right]_0^1$$

$$= \frac{1}{2}\left[\sqrt{5} + \frac{1}{4}\ln\left[\left(1+\frac{1}{8}\right) + \sqrt{\left(1+\frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2}\right]\right] - \frac{1}{8}\ln\frac{1}{8} \approx \mathbf{1.4789428575446}$$